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Research Paper

The reduction technique of singular equivalences of Morita type with level via Morita context algebras

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ABSTRACT

In the paper, we provide enough singular equivalence of Morita type with level arising from Morita context algebras inspired by the work of Gao-Zhao and Wang. Based on this, we establish an equivalence between their stable categories of Gorenstein-projective modules.

1. Introduction

Singularity category was first introduced by Buchweitz in the study of algebraic representations of Gorenstein rings in [5]. More precisely, for a finite-dimensional algebra A over a field k, the singularity category $D_{sg}(A)$ is the Verdier quotient $D^b(A)/K^b(A\text{-proj})$, where $K^b(A\text{-proj})$ is the full subcategory consisting of perfect complexes over A. Moreover, he showed that $D_{sg}(A)$ is equivalent to the stable category of finitely generated

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Gorenstein-projective modules $A\text{-}\underline{\mathrm{Gproj}}$ as triangulated categories when A is Gorenstein. Later, Orlov [24] rediscovered this notion from the perspective of algebraic geometry and mathematical physics, which has a deep relationship with Homological Mirror Symmetry.

In recent years, singularity categories and singular equivalences (i.e. the triangulated equivalence of singularity categories) have been extensively studied from various perspective and significant progress has been made in this field, such as in tilting theory ([6,17,22] etc.), homological algebra ([5,7,20,28,29] etc.), algebraic geometry ([3,18,31] etc.) and even in knot theory ([21]). Among them, singular equivalences of Morita type introduced by [7], which is analogous to the notion of stable equivalences of Morita type ([4]), have aroused some research interest. For example, Gao-Zhao [15] provided a new way to construct the singular equivalence of Morita type arising from Morita context algebras. Zhou-Zimmermann [32] showed that under some conditions singular equivalences of Morita type have some biadjoint functor properties and preserve positive degree Hochschild homology.

Not long after this, Wang [30] introduced a generalized version of singular equivalences of Morita type called singular equivalence of Morita type with level, and proved that this equivalence can be induced by a derived equivalence of standard type. This relatively new concept quickly attracted some attention, see, for instance, the articles [8,10,25–27,29].

Motivated by the following work: the study of the singular equivalence of Morita type with level between an algebra and certain upper triangular matrix algebra involved this algebra in [30], and the study of the singular equivalence of Morita type arising from Morita context algebras in [15]. It is natural to ask the question whether the constructable way is applicable to the setting of singular equivalence of Morita type with level.

In the paper, we give a positive answer for the above question. We will provide a constructable way to show the singular equivalences of Morita type arising from Morita context algebras. In addition, we will establish an equivalence between the stable categories of Gorenstein-projective modules induced by the singular equivalence of Morita type with level between two Morita context algebras.

Our main results are the following.

Theorem. *Let A and B be finite-dimensional k-algebras which are singularly equivalent of Morita type with level $n+1$ induced by $({}_AM_B,\ {}_BN_A)$. Let $\Lambda = \begin{pmatrix} A & {}_AV_C \\ {}_CW_A & C \end{pmatrix}$ and $\Gamma = \begin{pmatrix} B & N\otimes_A V \\ W\otimes_A M & C \end{pmatrix}$ be the Morita context algebras such that Λ satisfies conditions in Setting 2.3.*

(1) *The pair of bimodules $(\Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ \Gamma e_1 \otimes_B N \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda))$ defines a singular equivalence of Morita type with level $n+1$ between Λ and Γ.*

(2) *Assume that left B-module $\mathrm{Hom}_A(M, A)$ and left A-module $\mathrm{Hom}_B(N, B)$ are of finite projective dimension. Then there are the following two equivalences*

$$\Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma) \otimes_\Gamma - :\ \Gamma\text{-}\underline{\mathrm{Gproj}} \longrightarrow \Lambda\text{-}\underline{\mathrm{Gproj}}$$

and

$$\Gamma e_1 \otimes_B N \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda) \otimes_\Lambda - : \ \Lambda\text{-}\underline{\mathrm{Gproj}} \longrightarrow \Gamma\text{-}\underline{\mathrm{Gproj}},$$

which are quasi-inverse to each other.

(3) *If A and B are Gorenstein algebras, then there is a triangle-equivalence $\Lambda\text{-}\underline{\mathrm{Gproj}} \cong \Gamma\text{-}\underline{\mathrm{Gproj}}$.*

2. Singular equivalence of Morita type with level in Morita context case

In this section, we show that the singular equivalences of Morita type with level can be constructed from Morita context algebras. Furthermore, given a singular equivalence of Morita type with level induced by two bimodules between two algebras, we show a constructable way to new singular equivalence of Morita type with level between two Morita context algebras arising from above bimodules.

Let k be a field and A a finite-dimensional k-algebra. Denote by A-mod the category of finitely generated left A-modules, and by $A\text{-}\underline{\mathrm{mod}}$ the stable module category of A-mod modulo projective left A-modules. The syzygy functor $\Omega : A\text{-}\underline{\mathrm{mod}} \longrightarrow A\text{-}\underline{\mathrm{mod}}$ is defined on the stable module category, and can be computed as the kernel $\Omega(M) = \mathsf{Ker}(P \twoheadrightarrow M)$ of any epimorphism from a projective left A-module P. For simplicity, we denote $\otimes_k$ by $\otimes$, and denote the enveloping algebra $A \otimes A^{\mathrm{op}}$ of A by A^e, where A^{op} is the opposite algebra of A.

A singular equivalence of Morita type with level n was introduced by Wang ([30]) in case k is a commutative algebra.

Definition 2.1. ([30, Definition 2.1]) Let k be a commutative algebra and let A and B be two k-algebras which are projective as k-modules. Let M and N be, respectively, an A-B-bimodule and a B-A-bimodule. We say that (M, N) defines a singular equivalence of Morita type with level n for some $n \in N$ if the following conditions are satisfied:

(1) M is finitely generated projective as a left A-module and as a right B-module,
(2) N is finitely generated projective as a left B-module and as a right A-module,
(3) $M \otimes_B N \cong \Omega^n_{A^e}(A)$ in $A^e\text{-}\underline{\mathrm{mod}}$ and $N \otimes_A M \cong \Omega^n_{B^e}(B)$ in $B^e\text{-}\underline{\mathrm{mod}}$.

For convenience, we recall the definition of Morita context rings. And we refer to [2] for the terminology of Morita context rings.

Definition 2.2. Let A and B be rings, ${}_BM_A$ a B-A-bimodule, ${}_AN_B$ an A-B-bimodule, $\phi : M \otimes_A N \longrightarrow B$ a B-bimodule map, and $\psi : N \otimes_B M \longrightarrow A$ an A-bimodule map, satisfying the following associativity conditions, $\forall m, m' \in M, \forall n, n' \in N$:

$$\phi(m \otimes n)m' = m\psi(n \otimes m') \qquad \text{and} \qquad n\phi(m \otimes n') = \psi(n \otimes m)n'.$$

A Morita context ring is $\Lambda_{(\phi,\psi)} := \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$, with componentwise addition, and multiplication

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix}\begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa'+\psi(n\otimes m') & an'+nb' \\ ma'+bm' & \phi(m\otimes n')+bb' \end{pmatrix},$$

which is an associative ring.

Throughout this paper, we consider the Morita context ring $\Lambda = \begin{pmatrix} A & {}_AV_C \\ {}_CW_A & C \end{pmatrix}$, which is an Artin algebra with $W \otimes_A V = 0 = V \otimes_C W$.

Setting 2.3. Let C be a finite-dimensional k-algebra which has finite projective dimension m_1 as C-C-bimodule, W a C-A-bimodule with finite projective dimension m_2 as a bimodule such that ${}_CW$ and W_A are projective, and V an A-C-bimodule with finite projective dimension m_3 as a bimodule such that ${}_AV$ and V_C are projective. Set $n := \max\{m_1, m_2, m_3\}+1$. Let $\Lambda = \begin{pmatrix} A & {}_AV_C \\ {}_CW_A & C \end{pmatrix}$ be a Morita context ring which is an Artin algebra, such that $W \otimes_A V = 0 = V \otimes_C W$. Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

We give an example of the finite dimensional algebra A satisfying Setting 2.3.

Example 2.4. ([9, Example 4.3]) Let Q be the quiver $1 \longrightarrow 2$ and $A = kQ$ with char$k \neq 2$. Then $e_1Ae_2 = 0$ and $e_2Ae_1 \cong k$. Put $M = Ae_2 \otimes_k e_1A$. Then ${}_AM$ and M_A are projective, and

$$\begin{aligned} M \otimes_A M &= (Ae_2 \otimes_k e_1A) \otimes_A (Ae_2 \otimes_k e_1A) \\ &= Ae_2 \otimes_k (e_1A \otimes_A Ae_2) \otimes_k e_1A \\ &= 0 \end{aligned}$$

Take $\Lambda = \begin{pmatrix} A & M \\ M & A \end{pmatrix}$. Then Λ satisfies Setting 2.3.

Remark 2.5. In Setting 2.3, the choice of the algebra C is not unique. For example, C can be taken as a homologically smooth algebra (see [19]), this is, it admits a finite resolution by finitely generated projective bimodules if as a bimodule. In all, the Λ^e-module that have finite projective dimension induced by the algebra C (or more precisely by the algebra C and bimodules ${}_CW_A$, ${}_AV_C$) is significant in the main theorem.

2.1. Some projective modules and functors

In this subsection, we are assuming Setting 2.3, then we have some projective modules and we can consider functors between bimodule categories.

Lemma 2.6. (1) $\Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda)$ *is projective as a left A-module and as a right Λ-module.*

(2) Λe_1 *is projective as a left Λ-module and as a right A-module.*

Proof. (1) Since $e_1\Lambda e_1 = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) \cong A$ as algebras, $e_1\Lambda$ is an A-Λ-bimodule. Then there are an isomorphism

$$\Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda) \cong \Omega^{n+1}_A(e_1\Lambda)$$

in A-$\underline{\mathrm{mod}}$, as projective $A\otimes\Lambda^{\mathrm{op}}$-modules are projective left A-modules and syzygies are independent of the projective resolutions. As left A-modules,

$$e_1\Lambda = \left(\begin{smallmatrix} A & V \\ 0 & 0 \end{smallmatrix}\right) \cong A \oplus V,$$

which means that $e_1\Lambda$ is a projective left A-module. Hence, $\Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda) \cong \Omega^{n+1}_A(e_1\Lambda) = 0$ in A-$\underline{\mathrm{mod}}$, and so $\Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda)$ is a projective left A-module. As right Λ-modules, we have $e_1\Lambda \oplus e_2\Lambda \cong \Lambda$. Then $e_1\Lambda$ is a projective right Λ-module, and therefore $\Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda)$ is projective as a left A-module and as a right Λ-module.

(2) As left Λ-modules, we have $\Lambda e_1 \oplus \Lambda e_2 \cong \Lambda$. Hence Λe_1 is a projective left Λ-module. As right A-modules, there are isomorphisms

$$\Lambda e_1 = \left(\begin{smallmatrix} A & 0 \\ W & 0 \end{smallmatrix}\right) \cong A \oplus W.$$

Since W_A is projective, we have that Λe_1 is a projective right A-module. Therefore, Λe_1 is projective as a left Λ-module and as a right A-module. □

Consider functors

$$\begin{aligned} \mathcal{F}:\ C\otimes A^{\mathrm{op}}\text{-mod} &\longrightarrow \Lambda^e\text{-mod} \\ X &\longmapsto \left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right), \\ \mathcal{G}:\ A\otimes C^{\mathrm{op}}\text{-mod} &\longrightarrow \Lambda^e\text{-mod} \\ Y &\longmapsto \left(\begin{smallmatrix} 0 & Y \\ 0 & 0 \end{smallmatrix}\right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}:\ C\otimes C^{\mathrm{op}}\text{-mod} &\longrightarrow \Lambda^e\text{-mod} \\ Z &\longmapsto \left(\begin{smallmatrix} 0 & V\otimes_C Z \\ Z\otimes_C W & Z \end{smallmatrix}\right). \end{aligned}$$

Lemma 2.7. *The functors $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ are fully faithful and preserve projective modules.*

Proof. We observe that

$$\mathcal{F} \cong \Lambda e_2 \otimes_C - \otimes_A e_1\Lambda, \quad \mathcal{G} \cong \Lambda e_1 \otimes_A - \otimes_C e_2\Lambda, \quad \mathcal{H} \cong \Lambda e_2 \otimes_C - \otimes_C e_2\Lambda.$$

Following the proof of Lemma 2.6, Λe_1 is projective as a left Λ-module and as a right A-module, Λe_2 is projective as a left Λ-module and as a right C-module, $e_1\Lambda$ is projective as a left A-module and as a right Λ-module, and $e_2\Lambda$ is projective as a left C-module and as a right Λ-module. Then we get that $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ preserve projective modules, and moreover, $\mathcal{F}$ and $\mathcal{G}$ are fully faithful.

For any $X_1, X_2 \in C \otimes A^{\mathrm{op}}$-mod and $Y_1, Y_2 \in A \otimes C^{\mathrm{op}}$-mod, there are the following isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\Lambda^e}(\begin{pmatrix} 0 & 0 \\ X_1 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 0 \\ X_2 & 0 \end{pmatrix}) &\cong \mathrm{Hom}_{C\otimes A^{\mathrm{op}}}((X_1, 0),\ (X_2, 0)) \\ &\cong \mathrm{Hom}_{C\otimes A^{\mathrm{op}}}(X_1,\ X_2),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_{\Lambda^e}((\begin{smallmatrix} 0 & Y_1 \\ 0 & 0 \end{smallmatrix}),\ (\begin{smallmatrix} 0 & Y_2 \\ 0 & 0 \end{smallmatrix})) &\cong \mathrm{Hom}_{A\otimes C^{\mathrm{op}}}((0, Y_1),\ (0, Y_2)) \\ &\cong \mathrm{Hom}_{A\otimes C^{\mathrm{op}}}(Y_1,\ Y_2).\end{aligned}$$

Therefore $\mathcal{F}$ and $\mathcal{G}$ are fully faithful. Define the functor

$$\begin{aligned}\mathcal{H}' :\ \Lambda^e\text{-mod} &\longrightarrow C^e\text{-mod} \\ Z' &\longmapsto C \otimes_\Lambda Z' \otimes_\Lambda C,\end{aligned}$$

and we obtain the isomorphism $\mathcal{H}' \circ \mathcal{H} \cong \mathrm{Id}$. Hence $\mathcal{H}$ is faithful. On the other hand, there is an epimorphism

$$\mathrm{Hom}_{C^e}(Z_1,\ Z_2) \twoheadrightarrow \mathrm{Hom}_{\Lambda^e}(\mathcal{H}(Z_1),\ \mathcal{H}(Z_2)),$$

and so $\mathcal{H}$ is fully-faithful. □

2.2. Singular equivalence of Morita type with level

Let k be a field. We will show the singular equivalence of Morita type with level arising from Morita context algebras.

Proposition 2.8. *Based on Setting 2.3, the pair of bimodules $(\Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda),\ \Lambda e_1)$ defines a singular equivalence of Morita type with level $n+1$ between A and Λ.*

Proof. Note that

$$\Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda) \otimes_\Lambda \Lambda e_1 \cong \Omega^{n+1}_{A^e}(A) \text{ in } A^e\text{-}\underline{\mathrm{mod}}$$

and

$$\Lambda e_1 \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda) \cong \Omega^{n+1}_{\Lambda^e}(\Lambda e_1 \Lambda) \text{ in } \Lambda^e\text{-}\underline{\mathrm{mod}}.$$

We claim that $\Omega^{n+1}_{\Lambda^e}(\Lambda e_1\Lambda) \cong \Omega^{n+1}_{\Lambda^e}(\Lambda)$ in Λ^e-$\underline{\mathrm{mod}}$. In fact, there is a decomposition of Λ^e as a Λ-Λ-bimodule as follows:

$$\begin{aligned}\Lambda^e &\cong (\Lambda e_1 \oplus \Lambda e_2) \otimes (e_1\Lambda \oplus e_2\Lambda)\\ &\cong \left(\begin{smallmatrix} A & 0\\ W & 0\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} A & V\\ 0 & 0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} A & 0\\ W & 0\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 0\\ W & C\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & V\\ 0 & C\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} A & V\\ 0 & 0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & V\\ 0 & C\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 0\\ W & C\end{smallmatrix}\right).\end{aligned}$$

Then we have the following commutative diagram in Λ^e-mod

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \Omega^1_{\Lambda^e}(\left(\begin{smallmatrix} A & V\\ W & 0\end{smallmatrix}\right)) \oplus \Omega^1_{\Lambda^e}(\left(\begin{smallmatrix} 0 & V\\ W & C\end{smallmatrix}\right)) \oplus K & \longrightarrow & \Omega^1_{\Lambda^e}(\Lambda) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \left(\begin{smallmatrix} A & 0\\ W & 0\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} A & V\\ 0 & 0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & V\\ 0 & C\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 0\\ W & C\end{smallmatrix}\right) \oplus K & \xrightarrow{\cong} & \Lambda \otimes \Lambda^{\mathrm{op}} & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \left(\begin{smallmatrix} 0 & V\\ W & 0\end{smallmatrix}\right) & \longrightarrow & \left(\begin{smallmatrix} A & V\\ W & 0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & V\\ W & C\end{smallmatrix}\right) & \longrightarrow & \Lambda & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with $K = \left(\begin{smallmatrix} A & 0\\ W & 0\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 0\\ W & C\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & V\\ 0 & C\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} A & V\\ 0 & 0\end{smallmatrix}\right)$. From the Snake Lemma, we have an exact sequence

$$0 \longrightarrow \Omega^1_{\Lambda^e}(\left(\begin{smallmatrix} A & V\\ W & 0\end{smallmatrix}\right)) \oplus \Omega^1_{\Lambda^e}(\left(\begin{smallmatrix} 0 & V\\ W & C\end{smallmatrix}\right)) \oplus K \longrightarrow \Omega^1_{\Lambda^e}(\Lambda) \longrightarrow \left(\begin{smallmatrix} 0 & V\\ W & 0\end{smallmatrix}\right) \longrightarrow 0$$

in Λ^e-mod. Then we have a distinguished triangle

$$\Omega^1_{\Lambda^e}(\left(\begin{smallmatrix} 0 & V\\ W & 0\end{smallmatrix}\right)) \longrightarrow \Omega^1_{\Lambda^e}(\left(\begin{smallmatrix} A & V\\ W & 0\end{smallmatrix}\right)) \oplus \Omega^1_{\Lambda^e}(\left(\begin{smallmatrix} 0 & V\\ W & C\end{smallmatrix}\right)) \longrightarrow \Omega^1_{\Lambda^e}(\Lambda) \longrightarrow \left(\begin{smallmatrix} 0 & V\\ W & 0\end{smallmatrix}\right)$$

in the left triangulated category Λ^e-$\underline{\mathrm{mod}}$ since K is projective as a Λ-Λ-bimodule. Applying the shift functor to the distinguished triangle, we have the following distinguished triangle

$$\Omega^{n+1}_{\Lambda^e}(\left(\begin{smallmatrix} 0 & V\\ W & 0\end{smallmatrix}\right)) \longrightarrow \Omega^{n+1}_{\Lambda^e}(\left(\begin{smallmatrix} A & V\\ W & 0\end{smallmatrix}\right)) \oplus \Omega^{n+1}_{\Lambda^e}(\left(\begin{smallmatrix} 0 & V\\ W & C\end{smallmatrix}\right)) \longrightarrow \Omega^{n+1}_{\Lambda^e}(\Lambda) \longrightarrow \Omega^{n}_{\Lambda^e}\left(\begin{smallmatrix} 0 & V\\ W & 0\end{smallmatrix}\right).$$

Since W has finite projective dimension in $C \otimes A^{\mathrm{op}}$-mod, V has finite projective dimension in $A \otimes C^{\mathrm{op}}$-mod, and C has finite projective dimension in C^e-mod, it follows from Lemma 2.7 that $\mathcal{F}(W)$, $\mathcal{G}(V)$ and $\mathcal{H}(C)$ all have finite projective dimension smaller than n in Λ^e-mod. Hence there are the following isomorphisms in Λ^e-$\underline{\mathrm{mod}}$

$$\Omega^n_{\Lambda^e}\left(\begin{smallmatrix} 0 & 0 \\ W & 0 \end{smallmatrix}\right) \cong 0, \qquad \Omega^n_{\Lambda^e}\left(\begin{smallmatrix} 0 & V \\ 0 & 0 \end{smallmatrix}\right) \cong 0, \qquad \Omega^n_{\Lambda^e}(\left(\begin{smallmatrix} 0 & V \\ W & C \end{smallmatrix}\right)) \cong 0.$$

On the other hand, we can get from the exact sequence

$$0 \longrightarrow \left(\begin{smallmatrix} 0 & 0 \\ W & 0 \end{smallmatrix}\right) \longrightarrow \left(\begin{smallmatrix} 0 & V \\ W & 0 \end{smallmatrix}\right) \longrightarrow \left(\begin{smallmatrix} 0 & V \\ 0 & 0 \end{smallmatrix}\right) \longrightarrow 0$$

and the Horseshoe Lemma that $\Omega^n_{\Lambda^e}\left(\begin{smallmatrix} 0 & V \\ W & 0 \end{smallmatrix}\right) \cong 0$ in Λ^e-$\underline{\mathrm{mod}}$. Therefore we have $\Omega^{n+1}_{\Lambda^e}(\Lambda e_1 \Lambda) = \Omega^{n+1}_{\Lambda^e}(\left(\begin{smallmatrix} A & V \\ W & 0 \end{smallmatrix}\right)) \cong \Omega^{n+1}_{\Lambda^e}(\Lambda)$ in Λ^e-$\underline{\mathrm{mod}}$ from the last distinguished triangle above. Combined with Lemma 2.6, we show that $(\Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda),\ \Lambda e_1)$ defines a singular equivalence of Morita type with level $n+1$ between A and Λ. □

Remark 2.9. The condition "$W \otimes_A V = 0 = V \otimes_C W$" in Proposition 2.8 can not be weakened as "$\phi = 0 = \psi$". Otherwise, for example in the commutative diagram in the proof,

$$\left(\begin{smallmatrix} A & 0 \\ W & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} A & V \\ 0 & 0 \end{smallmatrix}\right) \longrightarrow \left(\begin{smallmatrix} A & V \\ W & 0 \end{smallmatrix}\right) \longrightarrow 0$$

is no longer a projective resolution; compare [9, Example 3.4, Remark 4.10]. We emphasize the difference with [10, Example 4.6].

Theorem 2.10. *Let A and B be finite-dimensional k-algebras which are singularly equivalent of Morita type with level $n+1$ induced by $({}_AM_B,\ {}_BN_A)$. Let $\Lambda = \begin{pmatrix} A & {}_AV_C \\ {}_CW_A & C \end{pmatrix}$ and $\Gamma = \begin{pmatrix} B & N\otimes_A V \\ W\otimes_A M & C \end{pmatrix}$ be the Morita context algebras such that Λ satisfies conditions in Setting 2.3. Then the pair of bimodules*

$$(\Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ \Gamma e_1 \otimes_B N \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda))$$

defines a singular equivalence of Morita type with level $n+1$ between Λ and Γ.

Proof. It is clear that ${}_CW \otimes_A M$, $W \otimes_A M_B$, ${}_BN \otimes_A V$ and $N \otimes_A V_C$ are projective, $W \otimes_A M$ has finite projective dimension m_2 as C-B-bimodule, and $N \otimes_A V$ has finite projective dimension m_3 as B-C-bimodule. It follows from Proposition 2.8 that the pair of bimodules $(\Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ \Gamma e_1)$ defines a singular equivalence of Morita type with level $n+1$ between B and Γ, this is, there are isomorphisms

$$\Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma) \otimes_\Gamma \Gamma e_1 \cong \Omega^{n+1}_{B^e}(B) \text{ in } B^e\text{-}\underline{\mathrm{mod}}$$

and

$$\Gamma e_1 \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma) \cong \Omega^{n+1}_{\Gamma^e}(\Gamma) \text{ in } \Gamma^e\text{-}\underline{\mathrm{mod}}.$$

Since $({}_AM_B,\ {}_BN_A)$ defines a singular equivalence of Morita type with level $n+1$ between A and B, there are isomorphisms

$$M \otimes_B N \cong \Omega^{n+1}_{A^e}(A) \text{ in } A^e\text{-}\underline{\text{mod}} \quad \text{and} \quad N \otimes_A M \cong \Omega^{n+1}_{B^e}(B) \text{ in } B^e\text{-}\underline{\text{mod}}.$$

Then we have that in Λ^e-$\underline{\text{mod}}$

$$\begin{aligned} &\Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\text{op}}}(e_1\Gamma) \otimes_\Gamma \Gamma e_1 \otimes_B N \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\text{op}}}(e_1\Lambda) \\ \cong\ &\Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B^e}(B) \otimes_B N \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\text{op}}}(e_1\Lambda) \\ \cong\ &\Lambda e_1 \otimes_A \Omega^{n+1}_{A^e}(A) \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\text{op}}}(e_1\Lambda) \\ \cong\ &\Omega^{n+1}_{\Lambda^e}(\Lambda) \end{aligned}$$

and in Γ^e-$\underline{\text{mod}}$

$$\begin{aligned} &\Gamma e_1 \otimes_B N \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\text{op}}}(e_1\Lambda) \otimes_\Lambda \Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\text{op}}}(e_1\Gamma) \\ \cong\ &\Gamma e_1 \otimes_B N \otimes_A \Omega^{n+1}_{A^e}(A) \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\text{op}}}(e_1\Gamma) \\ \cong\ &\Gamma e_1 \otimes_B \Omega^{n+1}_{B^e}(B) \otimes_B \Omega^{n+1}_{A\otimes\Lambda^{\text{op}}}(e_1\Lambda) \\ \cong\ &\Omega^{n+1}_{\Gamma^e}(\Gamma). \end{aligned}$$

On the other hand, it is clear that $\Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\text{op}}}(e_1\Gamma)$ is projective as a left Λ-module and as a right Γ-module, and $\Gamma e_1 \otimes_B N \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\text{op}}}(e_1\Lambda)$ is projective as a left Γ-module and as a right Λ-module. Therefore, we have that the pair of bimodules

$$(\Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\text{op}}}(e_1\Gamma),\ \Gamma e_1 \otimes_B N \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\text{op}}}(e_1\Lambda))$$

defines a singular equivalence of Morita type with level $n+1$ between Λ and Γ. □

Corollary 2.11. *Let $\Lambda = \left(\begin{smallmatrix} A & V \\ 0 & C \end{smallmatrix}\right)$ and $\Gamma = \left(\begin{smallmatrix} B & N\otimes_A V \\ 0 & C \end{smallmatrix}\right)$ be the upper triangular matrix algebras with C being a finite-dimensional k-algebra which has finite projective dimension m_1 as C-C-bimodule and V being an A-C-bimodule with finite projective dimension m_2 as a bimodule. Set $n := \max\{m_1, m_2\}+1$. If A and B are singularly equivalent of Morita type with level $n+1$ induced by $({}_AM_B,\ {}_BN_A)$, then the pair of bimodules*

$$(\Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\text{op}}}(e_1\Gamma),\ \Gamma e_1 \otimes_B N \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\text{op}}}(e_1\Lambda))$$

defines a singular equivalence of Morita type with level $n+1$ between Λ and Γ.

Proof. It follows from [30, Section 3] that the pair of bimodules $(\Omega^{n+1}_{A\otimes\Lambda^{\text{op}}}(e_1\Lambda),\ \Lambda e_1)$ defines a singular equivalence of Morita type with level $n+1$ between A and Λ. Similar to the proof of Theorem 2.10, we have the conclusion. □

3. Restriction on the stable categories of Gorenstein-projective modules

In this section, we show that a singular equivalence of Morita type with level between two Morita context algebras can induce an equivalence between their stable categories of Gorenstein-projective modules.

For the finite-dimensional k-algebra A, we denote by A-mod the category of finitely generated A-modules and by A-proj the full subcategory of all projective A-modules. Denoted by $K^b(A\text{-mod})$, $D(A\text{-mod})$ and $D^b(A\text{-mod})$ the bounded homotopy category, derived category and bounded derived category of A-mod, respectively.

Recall from [11] and [12, Proposition 10.2.6] that a module G in A-mod is Gorenstein-projective if there is an exact sequence of projective A-modules

$$\cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots,$$

which stays exact after applying $\mathrm{Hom}_A(-,P)$ for each projective A-module P, such that $G \cong \mathsf{Ker}\, d^0$. Denote by A-Gproj the full subcategory of all Gorenstein projective A-modules. Observe that $A\text{-proj} \subset A\text{-Gproj}$, and we denote by $A\text{-}\underline{\text{Gproj}}$ the stable category of A-Gproj that modulo A-proj.

We list more results about functors and module categories over the Morita context algebras $\Lambda = \begin{pmatrix} A & {}_AV_C \\ {}_CW_A & C \end{pmatrix}$ and $\Gamma = \begin{pmatrix} B & N\otimes_A V \\ W\otimes_A M & C \end{pmatrix}$ which will be used in the sequel.

Remark 3.1. (1) ([13, Proposition 2.4]) There are the following recollements of module categories:

$$\xymatrix{C\text{-mod} \ar[r]^{\mathrm{Z}_C} & \Lambda\text{-mod} \ar@/_1pc/[l]_{\mathrm{Q}_C} \ar@/^1pc/[l]^{\mathrm{P}_C} \ar[r]^{\mathrm{U}_A} & A\text{-mod}, \ar@/_1pc/[l]_{\mathrm{T}_A} \ar@/^1pc/[l]^{\mathrm{H}_A}} \qquad \xymatrix{A\text{-mod} \ar[r]^{\mathrm{Z}_A} & \Lambda\text{-mod} \ar@/_1pc/[l]_{\mathrm{Q}_A} \ar@/^1pc/[l]^{\mathrm{P}_A} \ar[r]^{\mathrm{U}_C} & C\text{-mod} \ar@/_1pc/[l]_{\mathrm{T}_C} \ar@/^1pc/[l]^{\mathrm{H}_C}}$$

such that

(i) $(\mathrm{T}_A, \mathrm{U}_A, \mathrm{H}_A)$ and $(\mathrm{T}_C, \mathrm{U}_C, \mathrm{H}_C)$ are adjoint triples.
(ii) The functors T_A, H_A, T_C, H_C are fully faithful.
(iii) $\mathsf{Ker}\,\mathrm{U}_A = C\text{-mod}$, $\mathsf{Ker}\,\mathrm{U}_C = A\text{-mod}$,

and

$$\xymatrix{C\text{-mod} \ar[r]^{\mathrm{Z}'_C} & \Gamma\text{-mod} \ar@/_1pc/[l]_{\mathrm{Q}'_C} \ar@/^1pc/[l]^{\mathrm{P}'_C} \ar[r]^{\mathrm{U}_B} & B\text{-mod}, \ar@/_1pc/[l]_{\mathrm{T}_B} \ar@/^1pc/[l]^{\mathrm{H}_B}} \qquad \xymatrix{B\text{-mod} \ar[r]^{\mathrm{Z}_B} & \Gamma\text{-mod} \ar@/_1pc/[l]_{\mathrm{Q}_B} \ar@/^1pc/[l]^{\mathrm{P}_B} \ar[r]^{\mathrm{U}'_C} & C\text{-mod} \ar@/_1pc/[l]_{\mathrm{T}'_C} \ar@/^1pc/[l]^{\mathrm{H}'_C}}$$

such that

(i) $(\mathrm{T}_B, \mathrm{U}_B, \mathrm{H}_B)$ and $(\mathrm{T}'_C, \mathrm{U}'_C, \mathrm{H}'_C)$ are adjoint triples.
(ii) The functors $\mathrm{T}_B,\ \mathrm{H}_B,\ \mathrm{T}'_C,\ \mathrm{H}'_C$ are fully faithful.
(iii) $\mathsf{Ker}\,\mathrm{U}_B = C\text{-mod},\ \mathsf{Ker}\,\mathrm{U}'_C = B\text{-mod}$.

(2) Observe that $\mathrm{T}_A := \Lambda e_1 \otimes_A -,\ \mathrm{H}_A := \mathrm{Hom}_A(e_1\Lambda, -),\ \mathrm{T}_B := \Gamma e_1 \otimes_B -$ and $\mathrm{H}_B := \mathrm{Hom}_B(e_1\Gamma, -)$.

We continue now with a result of an equivalence between A-$\underline{\mathrm{Gproj}}$ and B-$\underline{\mathrm{Gproj}}$ induced by the singular equivalence of Morita type with level between two k-algebras A and B, which motivated our work in this section.

Lemma 3.2. ([30, Proposition 4.5]) *Let $({}_AM_B,\ {}_BN_A)$ define a singular equivalence of Morita type of level n between A and B, such that $\mathrm{Hom}_A(M, A)$, as a left B-module, and $\mathrm{Hom}_B(N, B)$, as a left A-module, are of finite projective dimension. Then there are the following two equivalences*

$$M \otimes_B - : B\text{-}\underline{\mathrm{Gproj}} \longrightarrow A\text{-}\underline{\mathrm{Gproj}} \quad \textit{and} \quad N \otimes_A - : A\text{-}\underline{\mathrm{Gproj}} \longrightarrow B\text{-}\underline{\mathrm{Gproj}},$$

which are quasi-inverse to each other.

To establish the equivalence between Λ-$\underline{\mathrm{Gproj}}$ and Γ-$\underline{\mathrm{Gproj}}$ induced by the singular equivalence of Morita type with level between the Morita context algebras $\Lambda = \begin{pmatrix} A & {}_AV_C \\ {}_CW_A & C \end{pmatrix}$ and $\Gamma = \begin{pmatrix} B & N\otimes_A V \\ W\otimes_A M & C \end{pmatrix}$, we need the following lemma.

Lemma 3.3. *We have $\mathrm{Hom}_B(e_1\Gamma, B) \in K^b(\Gamma\text{-proj})$.*

Proof. By Remark 3.1, we have that the functors $\mathrm{T}_B,\ \mathrm{U}_B$ and H_B are exact. Then there is an adjoint triple of triangulated functors of derived categories

$$D(\Gamma\text{-Mod}) \begin{array}{c} \xleftarrow{\ \mathrm{T}_B\ } \\ \xrightarrow{\ \mathrm{U}_B\ } \\ \xleftarrow[\ \mathrm{H}_B\ ]{} \end{array} D(B\text{-Mod}),$$

where the induced derived functors are still denoted by $\mathrm{T}_B,\ \mathrm{U}_B$ and H_B. Since U_B preserves $K^b(\mathrm{proj})$, then H_B admits a right adjoint $\mathcal{R}$; see [23, Theorems 5.1 and 4.1]. For any $X^\bullet \in D^b(\Gamma\text{-mod})$, we have

$$\begin{aligned} \mathcal{R}(X^\bullet) &\cong \mathrm{Hom}_B(B,\ \mathcal{R}(X^\bullet)) \\ &\cong \mathrm{Hom}_\Gamma(\mathrm{H}_B(B),\ X^\bullet) \in D^b(B\text{-mod}). \end{aligned}$$

That is, the functor $\mathcal{R}$ preserves $D^b(\text{mod})$. Therefore following [1, Lemma 2.7], we have that H_B preserves $K^b(\text{proj})$, and hence

$$\mathrm{Hom}_B(e_1\Gamma, B) \simeq \mathrm{H}_B(B) \in K^b(\Gamma\text{-proj}). \quad \square$$

Proposition 3.4. *Let A and B be finite-dimensional k-algebras which are singularly equivalent of Morita type with level $n+1$ induced by $({}_AM_B,\ {}_BN_A)$, such that left B-module $\mathrm{Hom}_A(M, A)$ and left A-module $\mathrm{Hom}_B(N, B)$ are of finite projective dimension. Let $\Lambda = \begin{pmatrix} A & {}_AV_C \\ {}_CW_A & C \end{pmatrix}$ and $\Gamma = \begin{pmatrix} B & N\otimes_A V \\ W\otimes_A M & C \end{pmatrix}$ be the Morita context algebras such that Λ satisfies conditions in Setting 2.3. Then there are the following two equivalences*

$$\Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma) \otimes_\Gamma - :\ \Gamma\text{-}\underline{\mathrm{Gproj}} \longrightarrow \Lambda\text{-}\underline{\mathrm{Gproj}}$$

and

$$\Gamma e_1 \otimes_B N \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda) \otimes_\Lambda - :\ \Lambda\text{-}\underline{\mathrm{Gproj}} \longrightarrow \Gamma\text{-}\underline{\mathrm{Gproj}},$$

which are quasi-inverse to each other.

Proof. Following Lemma 3.2, we only need to show that

$$\mathrm{Hom}_\Lambda(\Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ \Lambda)$$

and

$$\mathrm{Hom}_\Gamma(\Gamma e_1 \otimes_B N \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda),\ \Gamma)$$

are of finite projective dimension as a left Γ-module and as a left Λ-module respectively.

We first show that $\mathrm{Hom}_\Lambda(\Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ \Lambda)$ has finite projective dimension as a left Γ-module. There are the following isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_\Lambda(\Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ \Lambda)\\
&= \mathrm{Hom}_\Lambda(\mathrm{T}_A(M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma)),\ \Lambda)\\
&\cong \mathrm{Hom}_A(M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ \mathrm{U}_A(\Lambda))\\
&= \mathrm{Hom}_A(M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ A \oplus V)\\
&\cong \mathrm{Hom}_A(M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ A) \oplus \mathrm{Hom}_A(M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ V),
\end{aligned}$$

where the first isomorphism holds by the adjoint pair $(\mathrm{T}_A, \mathrm{U}_A)$. Since ${}_AV$ is projective, we focus on the left Γ-module $\mathrm{Hom}_A(M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ A)$, and then we have the following isomorphisms

$$\begin{aligned}
\mathrm{Hom}_A(M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ A) &\cong \mathrm{Hom}_B(\Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ \mathrm{Hom}_A(M, A))\\
&\cong \mathrm{Hom}_B(\Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ B) \otimes_B \mathrm{Hom}_A(M, A).
\end{aligned}$$

Here, the first isomorphism is because of the Tensor-Hom adjoint pair, and the second isomorphism holds since $\mathrm{Hom}_A(M, A) \in K^b(B\text{-proj})$. We claim that

$$\mathrm{Hom}_B(\Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ B) \in K^b(\Gamma\text{-proj}).$$

Therefore, $\mathrm{Hom}_A(M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ A) \in K^b(\Gamma\text{-proj})$. For the claimation, consider the following isomorphisms

$$\begin{aligned}\mathrm{Hom}_B(\Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ B) &\cong \mathrm{Hom}_B(e_1\Gamma \otimes_\Gamma \Omega^{n+1}_{\Gamma^e}(\Gamma),\ B)\\ &\cong \mathrm{Hom}_\Gamma(\Omega^{n+1}_{\Gamma^e}(\Gamma),\ \mathrm{Hom}_B(e_1\Gamma, B))\\ &\cong \mathrm{Hom}_\Gamma(\Omega^{n+1}_{\Gamma^e}(\Gamma),\ \Gamma) \otimes_\Gamma \mathrm{Hom}_B(e_1\Gamma, B),\end{aligned}$$

where the third isomorphism holds since $\mathrm{Hom}_B(e_1\Gamma, B) \in K^b(\Gamma\text{-proj})$. Observe that $\mathrm{Hom}_\Gamma(\Omega^{n+1}_{\Gamma^e}(\Gamma),\ \Gamma) \in K^b(\Gamma\text{-proj})$. Therefore, the projective dimension of the left Γ-modules $\mathrm{Hom}_\Gamma(\Omega^{n+1}_{\Gamma^e}(\Gamma),\ \Gamma)$ and $\mathrm{Hom}_B(e_1\Gamma, B)$ are finite. Hence the left Γ-module $\mathrm{Hom}_B(\Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ B)$ has finite projective dimension, and so $\mathrm{Hom}_\Lambda(\Lambda e_1 \otimes_A M \otimes_B \Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ \Lambda)$ has finite projective dimension as a left Γ-module.

Similarly, we can prove that $\mathrm{Hom}_\Gamma(\Gamma e_1 \otimes_B N \otimes_A \Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda),\ \Gamma)$ is of finite projective dimension as a left Λ-module. Therefore we have the results. □

Remark 3.5. The singular equivalence in Proposition 2.8 can be restricted to the stable categories of Gorenstein projective modules. Indeed, $\mathrm{Hom}_\Lambda(\Lambda e_1, \Lambda) \cong A \oplus V$ is A-projective, and the proof of $\mathrm{Hom}_A(\Omega^{n+1}_{A\otimes\Lambda^{\mathrm{op}}}(e_1\Lambda),\ A) \in K^b(\Lambda\text{-proj})$ is completely similar to prove $\mathrm{Hom}_B(\Omega^{n+1}_{B\otimes\Gamma^{\mathrm{op}}}(e_1\Gamma),\ B) \in K^b(\Gamma\text{-proj})$.

It is known due to Buchweitz ([5]) and independently Happel ([16]) that the singularity categories of a Gorenstein algebra can be characterized by the stable category of its finitely generated Gorenstein-projective modules, this is, there is a triangle-equivalence $D_{sg}(A\text{-mod}) \cong A\text{-}\underline{\mathrm{Gproj}}$ provided that A is Gorenstein. We close this section with a corollary based on the criterion of when the Morita context algebra $\Lambda = \begin{pmatrix} A & {}_AV_C \\ {}_CW_A & C \end{pmatrix}$ is Gorenstein.

Corollary 3.6. *Let A and B be Gorenstein algebras which are singularly equivalent of Morita type with level $n+1$ induced by $({}_AM_B,\ {}_BN_A)$. Let $\Lambda = \begin{pmatrix} A & {}_AV_C \\ {}_CW_A & C \end{pmatrix}$ and $\Gamma = \begin{pmatrix} B & N\otimes_A V \\ W\otimes_A M & C \end{pmatrix}$ be the Morita context algebras such that Λ satisfies conditions in Setting 2.3. Then there is a triangle-equivalence $\Lambda\text{-}\underline{\mathrm{Gproj}} \cong \Gamma\text{-}\underline{\mathrm{Gproj}}$.*

Proof. Observe that there is an equivalence of categories

$$D_{sg}(\Lambda\text{-mod}) \cong D_{sg}(\Gamma\text{-mod})$$

by Theorem 2.10. Since A and B are Gorenstein algebras, we have that Λ and Γ are also Gorenstein algebras following [14, Corollary 4.6, Theorem 4.13]. Therefore, we have the equivalence $\Lambda\text{-}\underline{\mathrm{Gproj}} \cong \Gamma\text{-}\underline{\mathrm{Gproj}}$. $\square$

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Declaration of competing interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Data availability

No data was used for the research described in the article.

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